DOCTORAL THESIS:QUALITATIVE ANALYSIS ON SOME CLASSES OF NONLINEAR PROBLEMS

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The following Doctoral Thesis entitled *Qualitative methods in the study of some classes of nonlinear problems* has been approved by the committee under the programme POSDRU 2008-2013 under the supervision of Professor *Vicenţiu Rădulescu* at the University of Craiova.

This thesis contains two parts. The first one contains three chapters and it is dedicated to the theory of mean value theorems and their applications to linear integral operators. The second part contains three chapters and it is devoted to theory of variational-hemivariational inequalities and generalized variational-like inequalities.

In **Chapter 1** entitled **The development of some mean value theorems and the Volterra operator**, we present one of the main developments in the theory of mean value problems: *Flett's mean value theorem.* This result was proved by Flett in 1958 and asserts that for a real valued function on closed interval $[a, b]$, differentiable on (a, b), continuous on $[a, b]$, and $f'(a) = f'(b)$, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

This theorem as well as other of its extensions and generalizations will be used in the next two chapters in proving some mean value properties for some linear integral operators of Volterra type. On the other hand, in the second section of the chapter, we introduce the Volterra map, $V : L^2([0, 1]) \to L^2([0, 1])$ given by

$$Vf(x) := \int_0^x f(t)dt,$$

and we state and prove some of its basic properties.

Chapter 2 entitled **Mean value theorems for some linear integral operators** is based on the paper *Mean value theorems for some linear integral operators* published in *Electronic Journal of Differential Equations,* ***2009*** *(2009), no. 117, pp. 1–15* and written in collaboration with *Tudorel Lupu*. The paper was cited in:

- Y. Lu, N. Yang, Symbol error probability of QAM with MRC diversity in two-wave with diffuse power fading channels, *IEEE Communications Letters*, **15** (2011), 10–12.
- D. Cakmak, A. Tiryaki, Mean value theorem for holomorphic functions, *Eletronic Journal of Differential Equations*, **2012** (2012), No. 34, 1–6.

- O. Hutnik, J. Molnarova, Flett's mean value theorem: a survery, *preprint* (September 2013) at `arxiv.org/pdf/1309.5715.pdf`.

In this chapter, we study some mean value problems involving linear integral operators on the Banach space of continuous functions. The main tools used are are mean value theorems like Rolle and Flett in order to study existence of zeroes for linear integral operators like

$$T\phi(t) := \phi(t) - \int_0^t \phi(x)dx,$$

$$S\Psi(t) := t\Psi(t) - \int_0^t x\Psi(x)dx,$$

and

$$R\xi(t) := \xi(t) - \xi'(t)\int_0^t \xi(x)dx,$$

$$V\rho(t) := t\rho(t) - \rho'(t)\int_0^t x\rho(x)dx,$$

where $\phi, \Psi \in C([0,1])$ and $\xi, \rho \in C^1([0,1])$. The main results of the chapter assert that the following mean value problems

$$(1) \qquad \begin{cases} \int_0^1 f(x)dx \cdot Tg(c_1) = \int_0^1 g(x)dx \cdot Tf(c_1), \\ Tf(c_2) = Sf(c_2), \\ \int_0^1 f(x)dx \cdot Sg(\tilde{c_4}) = \int_0^1 g(x)dx \cdot Sf(\tilde{c_4}) \end{cases}$$

have solutions $c_1, c_2, \tilde{c_4} \in (0,1)$ for $f, g \in C([0,1])$,

$$(2) \qquad \begin{cases} \int_0^1 (1-x)f(x)dx \cdot Tg(c_7) = \int_0^1 (1-x)g(x)dx \cdot Tf(c_7), \\ \int_0^1 (1-x)f(x)dx \cdot Sg(\tilde{c_7}) = \int_0^1 g(1-x)(x)dx \cdot Sf(\tilde{c_7}) \end{cases}$$

have solutions $c_7, \tilde{c_7} \in (0,1)$ for $f, g \in C([0,1])$,

$$(3) \qquad \begin{cases} \int_0^1 f(x)dx \cdot Rg(c_3) = \int_0^1 g(x)dx \cdot Rf(c_3), \\ \int_0^1 f(x)dx \cdot Vg(\overline{c}_4) = \int_0^1 g(x)dx \cdot Vf(\overline{c}_4) \end{cases}$$

have solutions $c_3, \overline{c}_4 \in (0,1)$ for $f, g \in C^1([0,1])$,
and

$$(4)\qquad \begin{cases} \int_0^1 (1-x)f(x)dx \cdot Rg(c_8) = \int_0^1 (1-x)g(x)dx \cdot Rf(c_8), \\ \int_0^1 (1-x)f(x)dx \cdot Vg(\tilde{c_8}) = \int_0^1 (1-x)g(x)dx \cdot Vf(\tilde{c_8}) \end{cases}$$

have solutions $c_8, \tilde{c_8} \in (0,1)$ for $f, g \in C([0,1])$,

The main idea of the proof of the problem (0.1) is based on the construction of the following auxiliary functions:

$$h_1(t) = f(t)\int_0^1 g(x)dx - g(t)\int_0^1 f(x)dx,$$

and

$$h_2(t) = (t-1)f(t),$$

together with the facts:

- if $h_1 \in C([0,1])$ satisfies the condotion $\int_0^1 h_1(x)dx = 0$, then there exist $c_1, \tilde{c_4} \in (0,1)$ such that
$$h_1(c_1) = \int_0^{c_1} h_1(x)dx,$$
and
$$\tilde{c_4} h_1(\tilde{c_4}) = \int_0^{\tilde{c_4}} x h_1(x)dx.$$
- if $h_2 \in C([0,1])$ satisfies the condition $h_2(1) = 0$, then there exists $c_2 \in (0,1)$ such that
$$h_2(c_2) = \int_0^{c_2} h_2(x)dx,$$

In the case of problem (0.2) the procedure is similar, but the auxiliary function we construct is given by

$$h_8(t) = f(t)\int_0^1 (1-x)g(x)dx - g(t)\int_0^1 (1-x)f(x)dx,$$

and by applying the same mean value results as above, but the condition is $\int_0^1 h_8(x)dx = \int_0^1 x h_8(x)dx$. Problems (0.3) and (0.4) are treated similarly by constructing the the same auxiliary functions but employing different mean value problems.

Chapter 3 entitled **Mean value problems of Flett type for a Volterra operator** is mainly based on the paper *Mean value problems of Flett type for a Volterra operator* published in *Electronic Journal of Differential Equations, vol.* ***2009*** *(2009), No. 53, pp. 1–7.* However, section 3.2 contains a result about zeroes in the image of a Volterra map and it is based on the paper *Zeroes of functions in the image of a Volterra operator* published in *Gazeta Matematică, A-series,* ***27*** *(2009), 209–213* with *Radu Gologan.* The papers have been cited in:

- O. Hutnik, J. Molnarova, Flett's mean value theorem: a survery, *preprint* (September 2013) at `arxiv.org/pdf/1309.5715.pdf`.

In this chapter, we study mean value problems for pairs of integrals involving a Volterra map. The first problems we deal with is the following:

$$\int_0^1 f(x)dx \cdot V_\phi g(c) = \int_0^1 g(x)dx \cdot V_\phi f(c) \tag{5}$$

has a solution $c \in (0,1)$, where $f, g \in C([0,1])$ and $\phi \in C^1([0,1])$ satisfying certain conditions, where $V_\phi f(t) := \int_0^t \phi(x)f(x)dx$ is the Volterra map.

Problem (0.5) is treated in another paper under "minimal" assumptions for ϕ. In fact, it is shown that the mean value problem (0.5) has a solution $c \in (0,1)$ if $\phi : [0,1] \to \mathbb{R}$ is nondecreasing, continuous at 0, and $\phi(0) = 0$. On the other hand, in the paper of Mingarelli and Plaza it is assumed that $\phi \in C^1([0,1])$, $\phi(x) \geq 0$ and $\phi'(x) \geq 0$ for all $x \in (0,1)$.

The proof of this result is rather elementary, but tricky! The idea goes as it follows: consider $h : [0,1] \to \mathbb{R}$, given by $h(x) = f(x) - \lambda g(x)$ where $\lambda = \dfrac{\int_0^1 f(x)dx}{\int_0^1 g(x)dx}$. Now, problem (0.5) reduces to prove that there exists $c \in (0,1)$ such that

$$V_\phi h(c) = \int_0^c \phi(x)f(x)dx = 0,$$

provided that $\displaystyle\int_0^1 h(x)dx = 0$. In this sense, we denote $H : [0,1] \to \mathbb{R}$, given by

$$H(t) = V_\phi h(t) = \int_0^t \phi(x)h(x)dx.$$

We first show that $\displaystyle\lim_{t\to 0^+} \frac{H(t)}{\phi(t)} = 0$ and the argument continues with integration by parts in the Riemann-Stieltjes setting followed by the intermediate value property. In the next section of this chapter we prove more general results than (0.5). The first main result is that the mean value problem

$$V_g f(c) = g(a) \cdot V f(c) \tag{6}$$

has a solution $c \in (a,b)$ where $f \in C_{\text{null}}([a,b])$ and $g \in C^1([a,b])$, and $g'(x) \neq 0$ for all $x \in [a,b]$. The main idea of the proof is to construct the following functions:

$$u(t) = \int_a^t f(x)g(x)dx - g(t)\int_a^t f(x)dx,$$

and

$$v(t) = g(t), t \in [a,b].$$

By an extension of Flett's mean value theorem, the conclusion follows immediately. On the other hand, the second main result states that the mean value problem

(7)
$$V_\phi f(x_0)\int_0^1 g(x)dx - V_\phi g(x_0)\int_0^1 f(x)dx = \phi(0)\left(Vf(x_0)\int_0^1 g(x)dx - Vg(x_0)\int_0^1 f(x)dx\right)$$
has a solution $x_0 \in (0,1)$ for $f, g \in C([0,1])$.

As in the proof of (0.6) the idea is to construct the following auxiliary functions:

$$u(t) = (\phi(t)Vf(t) - V_\phi f(t))\int_0^1 g(x)dx - (\phi(t)Vg(t) - V_\phi g(t))\int_0^1 f(x)dx$$

and

$$v(t) = \phi(t), t \in [0,1].$$

By the same extension of Flett's mean value theorem as in (0.6) we obtain the conclusion. As a particular case, if $\phi(0) = 0$, we obtain problem (0.5).

Chapter 4 entitled **Variational, Variational-Hemivariational and Variational-like Inequalities. A brief review.** introduces us to the theory of variational, variational-hemivariational inequalities and variational-like inequalities.

Firstly, we present some basic, but important tools from Convex and Nonsmooth Analysis like: semicontinuous and convex functionals in Banach spaces, convex and Lipschitz functions, Clarke's generalized directional derivative, etc.

Secondly, we review some important facts about monotone operators and related results that will be used in the next two chapters in proving existence of solutions for some variational-hemivariational and variational-like inequality problems. The variational inequality problem is the problem of finding $u \in K$, where K is a subset of a Banach space X which satisfies some properties of compactness and convexity, such that

$$\langle Au, v-u\rangle \geq 0, \forall v \in K, \tag{8}$$

where $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ is the duality pairing.

In general, a variational inequality can be formulated in any finite or infinite dimensional spaces. The ida of variational inequalities were extended and generalized to another class, which is known nowdays as the theory of hemivariational inequalities. The hemivariational inequalities have been introduced and investigated by Panagiotopoulos. The problem is the following:

Find $u \in K$ such that given a nonlinear operator $A : H \to H$, where H is a real Hilbert space, we have

$$\langle Au, v-u\rangle + \int_\Omega f^0(x, u, v-u)d\Omega \geq 0, \forall v \in K. \tag{9}$$

Here $f^0(x,u,v-u)$ denotes the generalized directional derivative. Panagiotopoulos studied these kind of inequalities in order to formulate variational principles associated with energy functions which are not neither convex nor smooth. If $f = 0$, then problem (0.9) becomes the classical variational inequality (0.8).

In 1989, Parida, Sahoo and Kumar proposed another type of variational inequality, namely *variational-like inequality* which is devoted to the following problem:

Find $u \in K$ such that for two continuous maps $A : K \to \mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$, we have

$$\langle Au, \eta(v,u)\rangle \geq 0, \forall v \in K. \tag{10}$$

If $\eta(u,v) = v - u$, then we obtain again the classical variational inequality problem (0.8).

The proofs of the existence of solutions for these variational, variational-hemivariational and variational-like inequality problems are based on some topological (fixed point) principles for set-valued mappings due to Knaster-Kuratowski-Mazurkiewicz, Ky Fan, Kakutani, Tarafdar, Mosco or Ansari-Yao. For instance, one of the most used tools is the celebrated Knaster-Kuratowski-Mazurkiewicz (KKM) topological principle obtained from the Sperner combinatorial lemma, and applied it to a simple proof of Brouwer's fixed point theorem,

KKM principle. *Let K be a nonempty subset of a Haussdorff topological vector space E and let $G : K \to 2^E$ be a set-valued mapping satisfying the following conditions:*

(i) G has the KKM property: for any $\{x_1, x_2 \ldots, x_n\} \subseteq K$, we have $\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$*, where* $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ *denotes the convex hull of* $\{x_1, x_2, \ldots, x_n\}$*;*

(ii) $g(x)$ is closed in E for every $x \in K$;

(iii) $G(x_0)$ is compact in E for some $x_0 \in K$.

Then

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

In **Chapter 5** entitled **Variational-Hemivariational inequalities involving set-valued mappings** is based on the paper *On a class of variational-hemivariational inequalities involving set valued mappings* published in *Advances in Pure and Applied Mathematics,* ***1*** *(2010), 233–246*, in collaboration with *Nicuşor Costea.* The paper has been cited in:

- Y. L. Zhang, Y. R. He, On stably quasimonotone hemivariational inequalities, *Nonlinear Analysis (TMA)* **74** (2011), 3324–3332.
- G. Tang, N-J. Huang, Existence theorems of the variational-hemivariational inequalities, *Journal of Global Optimization* **56** (2013), 605–622.
- Y. L. Zhang, Y. R. He, The hemivariational inequalities for an upper semicontinuous set-valued mapping, *Journal of Optimization Theory and Applications* **156** (2013), 716–725.
- R. Wankeeree, P. Preeschasilp, Existence theorems of the hemivariational inequality governed by a multi-valued map perturbed with a nonlineaer term in Banach spaces, *J. Global Optim.* **57** (2013), 1447–1464.

In this chapter, we establish some existence results for variational-hemivariational inequalities involving monotone set-valued mappings on bounded, closed and convex subsets in reflexive Banach spaces. We also derive several conditions for the existence of solutions in the case of unbounded subsets. The main goal of is to study the following problem:

Find $u \in K$ and $u^* \in A(u)$ such that

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0, \quad \forall v \in K, \tag{11}$$

which is closely related to the following dual problem

$$\sup_{v^* \in A(v)} \langle v^*, u-v\rangle \leq \phi(v) - \phi(u) + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx, \quad \forall v \in K. \tag{12}$$

Using the celebrated principle Knaster-Kuratowski-Mazurkiwievicz (KKM) and some other related topological results, we establish that if K is a non-empty, bounded, closed and convex subset of a real reflexive Banach space X, and $A : K \to 2^{X^*}$ is a set-valued mapping which is monotone, lower hemicontinuous on K, and if $T : X \to L^p(\Omega; \mathbb{R}^k)$ is linear and compact, and j satisfies the condition:

$$|z| \leq C(1 + |y|^{p-1})$$

almost everywhere $x \in w$, for all $y \in \mathbb{R}^k$ and $z \in \partial j(x, y)$, then the variational-hemivariational inequality (0.11) has at least one solution.

The main idea of the proof is to define $F, G : D(\varphi) \to 2^X$ as follows:

$$F(v) = \Big\{ u \in K \cap D(\phi) : \exists u^* \in A(u) \text{ such that } \langle u^*, v-u\rangle + \phi(v) - \phi(u) + \\ + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0 \Big\}$$

and

$$G(v) = \left\{ u \in K \cap D(\phi) : \sup_{v^* \in A(v)} \langle v^*, u-v\rangle \leq \phi(v) - \phi(u) + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \right\}.$$

The proof is divided into several steps, but the first goal is to show that the maps F and G are KKM followed by the fact that G is weakly compact for all $v \in K \cap D(\phi)$.

In the case when K is unbounded, let $u_n \in K_n := \{u \in K : ||u|| \leq n\}$ and $u_n^* \in A(u_n)$ be two sequences that satisfy

$$\langle u_n^*, v-u_n\rangle + \phi(v) - \phi(u_n) + \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x) - \hat{u}_n(x))dx \geq 0 \tag{13}$$

for all $v \in K_n$ and for every $n \geq 1$.

The second main result of the chapter asserts that under some hypothesis, the existence of solutions is guaranted for problem (0.11).

Chapter 6 entitled **Variational-like inequality problems involving set-valued maps and generalized monotonicity** is based on the paper *Variational-like inequality problems involving set-valued maps and generalized monotonicity* published in *Journal of Optimization Theory and Applications,* ***155*** *(2012), 79–99* in collaboration with *Nicuşor Costea* and *Daniel Alexandru Ion.* The paper has been recently cited in:

- S. Park, Recent applications of the Fan-KKM theorem, *Publications of the Research Institute for Mathematical Sciences* (RIMS-Kyoto), to appear.

The aim of this chapter is to establish existence results for some variational-like inequality problems involving set-valued maps, in reflexive and non-reflexive Banach spaces. When the set K, in which we seek for solutions, is compact and convex, we do not impose any monotonicity assumptions on the set-valued map A, which appears in the formulation of the inequality problems. In the case when K is only bounded, closed, and convex, certain monotonicity assumptions are needed: We ask A to be *relaxed* $\eta-\alpha$ *monotone* for generalized variational-like inequalities and *relaxed* $\eta-\alpha$ *quasimonotone* for variational-like inequalities. We also provide sufficient conditions for the existence of solutions in the case when K is unbounded, closed, and convex.

The main problems we study in this chapter are the following variational-like inequalities:

Find $u \in K \cap D(\phi)$ such that

$$\exists u^* \in A(u) : \langle u^*, \eta(v,u)\rangle + \phi(v) - \phi(u) \geq 0, \forall v \in K, \tag{14}$$

and

Find $u \in K$ such that

$$\exists u^* \in A(u) : \langle u^*, \eta(v,u)\rangle \geq 0, \forall v \in K, \tag{15}$$

where $K \subseteq X^{**}$ is nonempty, closed, and convex, $\eta : K \times K \to X^{**}$, $A \rightrightarrows X^*$ is a set-valued map and $\phi : X^{**} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional such that $K_\phi := K \cap D(\phi) \neq \emptyset$. Here $D(\phi)$ stands for the domain of the functional ϕ, i.e. $D(\phi) = \{u \in X^{**} : \phi(u) < +\infty\}$.

One of the main results of this chapter states that for K nonempty, bounded, closed, and convex subset of the real reflexive Banach space X, and for $A : K \rightrightarrows X^*$ *relaxed* $\eta - \alpha$ *monotone* map, under certain conditions, problem (0.14) admits at least one classical and strong solutions. The main idea of the proof is the use of the Mosco's alternative for the weak topology of X, and to define $\xi, \zeta : X \times X \to \mathbb{R}$,

$$\xi(v,u) = -\inf_{v^* \in A(v)} \langle v^*, \eta(v,u)\rangle + \alpha(v-u),$$

and

$$\zeta(v,u) = \sup_{u^* \in A(u)} \langle u^*, \eta(u,v)\rangle.$$

On the other hand, we also apply the celebrated KKM principle to derive classical and strong solutions to other variational-like inequality problems.

More exactly, if we weaken even more the hypothesis on the set-valued mapping $A : K \rightrightarrows X^*$ is *relaxed* $\eta - \alpha$ *quasimonotone*, the existence of solutions for problem (0.14) is still open. However, we prove that problem (0.15) admits, under certain hypothesis, at least one classical solutions and at least one strong solution. The main idea is to define the map $G : K \rightrightarrows X$ in the following way:

$$G(v) = \{u \in K : \langle v^*, \eta(v,u)\rangle \geq \alpha(v-u), \forall v^* A(v)\}.$$

The proof continues with the use of the KKM principle in order to obtain at least one strong solution for problem (0.15).